Unsteady Transonic Flow over Cascade Blades

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At present no analytical model is available for predicting the unsteady aerodynamic forces acting on staggered cascade blades subjected to transonic flow. The unsteady aerodynamic models for cascades developed so far are useful in the Mach number range of 0.0-0.9 and 1.1 and above. The objective of the present analysis is to develop an efficient model for obtaining unsteady aerodynamic forces in the neighborhood of Mach number = 1.0. An incremental annulus of blade row is represented by a rectilinear two-dimensional cascade of thin flat plate airfoils. The steady flow approaching the cascade is assumed to be transonic, irrotational, and inviscid. The equations of motion are derived using linearized transonic small perturbation theory. An analytical solution is obtained by using the Wiener-Hopf procedure. Unsteady aerodynamic forces and moments acting on the blades are obtained for Mach number = 1.0. Making use of transonic similarity law, the results of the present analysis are compared with the results obtained from other linearized cascade analyses. A parametric study is conducted to find the effects of reduced frequency, stagger angle, solidity, and location of pitching axis on cascade stability.

Nomenclature

a_n	= constant, Eq. (29)
A',B'	= constant, Eq. (A10)
A_0	= amplitude of angular displacement
A_1,B_1	= constants
b	= blade semichord
$C'(\alpha),C_-(\alpha)$	= functions, Eq. (43)
d_0	= distance between leading edge of blade and reference point
G_+,G_-	= functions, Eq. (A7)
$G_n^+,\bar{G}_+$	= functions, Eq. (42)
h_0,h_m	= functions, Eq. (18)
$h_n^{\pm},\bar{h}$	= functions, Eq. (38) and (39)
H_0	= amplitude of vertical displacement
$H_+(\alpha)$	= function, Eq. (A1)
i	$=\sqrt{-1}$
Im	= imaginary part
k	= reduced frequency
k_1	$=\sqrt{2/k}$
k_2	$=k/2$
$K(\alpha,\eta)$	= function, Eq. (23)
$K_+(\alpha),K_-(\alpha)$	= functions, Eq. (A5)
$K'_-(\alpha)$	$=1/K(\alpha)$
L	= nondimensional lift
m,n	= summation indices
M	= nondimensional moment
M_1	= local Mach number
P	= pressure
P_1',P_1'',P_n	= pressure for nonsummation and summation terms, Eq. (36)
Q_n	= function, Eq. (36)
s	= distance between adjacent blades
s^+	$=s/b\cos(\beta^*)$
s^-	$=s/b\sin(\beta^*)$
sgn	= signum function
t	= time
$T_1,\bar{T}_1,\bar{T}_2,\bar{\bar{T}}_2,\bar{\bar{T}}_1$	= functions, Eqs. (28-33)
U_∞	= freestream velocity
V,V_1,V_2	= upwash velocity
W_0	= blade displacement
x^*,x,y	= Cartesian coordinates
α	= complex coordinate
β^*	= stagger angle
γ_1	$=k_1\sqrt{\alpha+k_2}$
δ	= thickness ratio
Δ_+,Δ_-	= functions, Eq. (21)
η	$=ky/b$
η'	$=M_1\eta$
λ_1	= function, Eq. (20)
ξ	$=x/b$
ξ^*	$=\xi-2.0$
ρ_0	= density of fluid
σ	= interblade phase angle
τ	= nondimensional time
ϕ,ϕ_1,ϕ^*	= velocity potential
ω	= frequency of blade motion

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Introduction

THERE is a need for unsteady transonic airload prediction methods suitable for aeroelastic analysis of turbomachine blading. Considerable progress has been made in the development of cascade analysis for incompressible subsonic and supersonic flows. Whitehead[1] reported a method for calculating the aerodynamic forces and moments on unstalled vibrating cascade blades subjected to incompressible and inviscid flow. Smith[2] described a method for unsteady, subsonic flow through an infinite two-dimensional cascade of flat plate blades at zero incidence. A number of analyses, both semianalytical and numerical, have been performed where the basic flow is supersonic. Adamczyk and Goldstein[3] and Verdon[4] obtained a solution for unsteady flow in a supersonic cascade with subsonic leading-edge locus. By comparison, the progress on the development of such methods for transonic flow has been slow. Savkar[14] examined the problem of a thin airfoil oscillating in a transonic stream in a wind tunnel. This problem represents a zero staggered cascade whose members oscillate 180 deg out of phase with each other. Recently Verdon and Caspar[5] developed the unsteady transonic analysis for cascades of vibrating sharp-edged double circular arc airfoils. Kerlick and Nixon[6] used a high-frequency version of the Ballhaus and Goorjian[7] code LTRAN2 to represent unsteady aerodynamic phenomena in transonic cascade flow. Verdon and Caspar[5] and Kerlick and Nixon[6] made use of numerical methods for obtaining aerodynamic forces. A completely analytical solution for unsteady transonic flow over cascade blades is presented in this paper.

An incremental annulus of turbomachine stage is replaced by a rectilinear two-dimensional cascade of thin flat plate airfoils. The steady relative flow approaching the cascade is assumed to be transonic, irrotational, isentropic, and inviscid. The blades are allowed to undergo a small-amplitude harmonic oscillation, which generates a small unsteady flow superimposed on the existing steady flowfield. The blades are assumed to oscillate with a prescribed motion of constant amplitude and constant interblade phase angle. Since transonic fan and compressor blades are thin, it is assumed that steady flow deviates slightly from a uniform base flow. At large reduced frequencies, the steady and unsteady flowfields decouple under these conditions. Therefore, the unsteady flow can be calculated independently of the steady flow perturbations. The thickness, camber, and mean angle of attack of the blades only influence the steady flow perturbations. Therefore, for the purpose of this analysis, the blades can be replaced by a set of zero-thickness flat plates as shown in Fig. 1.

It is assumed that the disturbances are prohibited from propagating upstream in a transonic flow. Hence, the flow downstream of the Mach wave emanating from the trailing edge of each blade does not influence the flow upstream of this wave. We take advantage of this assumption to calculate the flow upstream of these Mach waves. This portion of the cascade may be replaced by a row of semi-infinite plates, as shown in Fig. 2. An analytical solution to the problem for this flow region is obtained using the Wiener-Hopf procedure.[11] The solution downstream of trailing-edge Mach wave is obtained by considering a backward-facing row of semi-infinite plates. It is assumed that the velocity component normal to the plates vanishes and that the jump in pressure across the wake region is equal and opposite to that given by the upstream solution. The analytical solution for this region is again obtained using the Wiener-Hopf procedure.[11] This solution is identically zero upstream of trailing-edge Mach waves. Hence, when the downstream solution is added to the upstream solution, an exact solution that satisfies all of the boundary conditions for an oscillating staggered cascade is obtained. This procedure was employed by Goldstein et al.[8] and Adamczyk et al.[3] to obtain the solution for supersonic flow with and without a strong in-passage shock.

Formulation

In the transonic regime, the appropriate representation of the flowfield for small disturbances is given by a nonlinear equation. The equation can be linearized when free Mach number is close to 1 and the reduced frequency k satisfies the requirements $k \gg |1-M_1|$ and $k \gg \delta^{2/3}$, where M_1 is the local Mach number and δ is the thickness-to-chord ratio. The linearization of the equation of motion has been extensively discussed by Landahl.[9] The unsteady small perturbation equation as derived by Landahl[9] for transonic flow is the field equation used in the present analysis. The blades are replaced by flat plates oscillating harmonically about their mean positions $\eta = kms^-$ for $m=0, \pm 1, \pm 2\ldots$. The unsteady wakes are assumed to be vortex sheets emanating from the trailing edge of the blades and having mean positions along the lines $\eta = mks^-$ for $m=0, \pm 1, \pm 2\ldots$. For the purpose of this analysis, the blades oscillate harmonically in time with the same amplitude and a constant but arbitrary interblade phase angle.

The mean flow is in the x direction. This flow is slightly disturbed by the blade that is also placed along the x axis. All the lengths are nondimensionalized by the blade semichord b, and time τ is nondimensionalized by multiplying physical time by U_∞/b.

The pressure fluctuation p' is nondimensionalized by the undisturbed freestream density ρ_0 multiplied by U_∞^2, and all fluctuating velocities are nondimensionalized by U_∞. Flow perturbations are assumed to be small. The governing equation considered below is based on the assumption that the flow is inviscid, irrotational, and isentropic. A perturbation velocity potential $U_\infty \bar{\phi} b$ is introduced to reduce the number of dependent variables. A small disturbance analysis, including the unique properties of transonic flows, yields in two dimensions

$$k^2 \frac{\partial^2 \bar{\phi}}{\partial \eta^2} - 2\frac{\partial^2 \bar{\phi}}{\partial \xi \partial \tau} - \frac{\partial^2 \bar{\phi}}{\partial \tau^2} = 0 \tag{1}$$

where

$$\xi = x/b, \quad \eta = k(y/b), \quad \tau = tU_\infty/b, \quad k = \omega b/U_\infty \tag{2}$$

Since the present problem is linear, all motion induced by the harmonic oscillation of the cascade blading must yield a harmonic time dependence for the potential.

$$\bar{\phi}(\xi,\eta,\tau) = \phi(\xi,\eta)e^{(-ik\tau)} \tag{3}$$

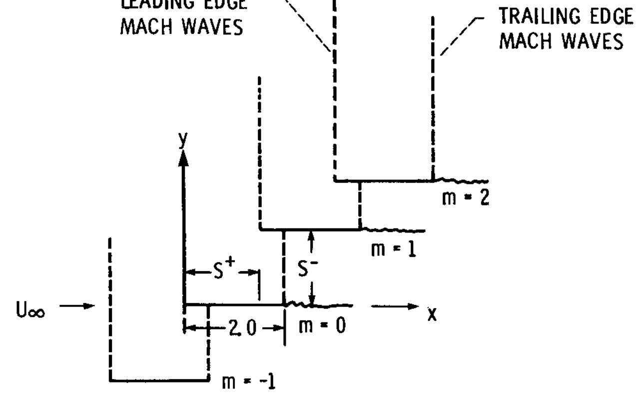

Fig. 1 Cascade configuration.

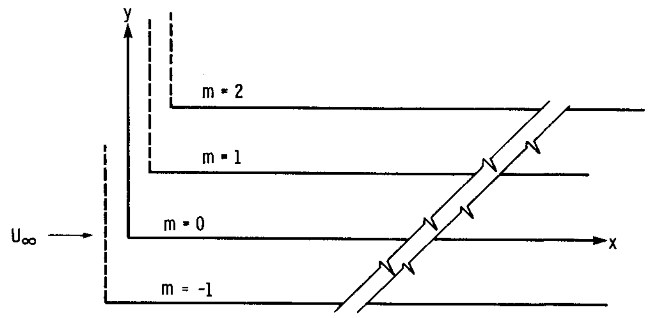

Fig. 2 Configuration for upstream solution.

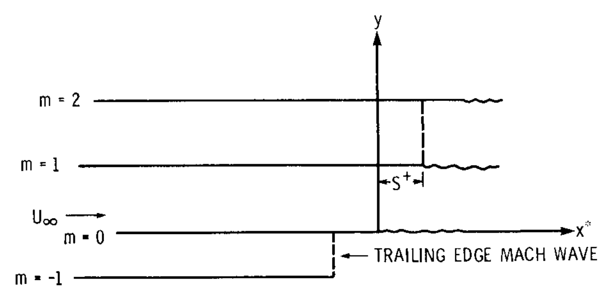

Fig. 3 Configuration for downstream solution.

Substitution of Eq. (3) into Eq. (1) yields

$$k^2\frac{\partial^2\phi}{\partial\eta^2}+2ik\frac{\partial\phi}{\partial\xi}+k^2\phi=0 \tag{4}$$

The amplitudes

$$P=p'e^{(ik\tau)} \qquad V=ve^{(ik\tau)} \tag{5}$$

of pressure fluctuation p' and upwash velocity fluctuation v can be determined from the following relations:

$$V=\frac{\partial\phi}{\partial\eta} \tag{6}$$

$$P=\left(ik-\frac{\partial}{\partial\xi}\right)\phi \tag{7}$$

The upwash velocity on the mth blade is assumed to differ from that on the zeroth blade only by a constant phase angle σ.

$$V(\xi+ms^+,mks^-\pm 0)=e^{(im\sigma)}V(\xi,\pm 0)$$
$$0<\xi<2;\quad m=0,\pm 1,\pm 2 \tag{8}$$

where $+0$ denotes the limit as $\eta\to 0$ from above and -0 denotes the limit as $\eta\to 0$ from below. This equation determines the upwash velocity on the mth blade in terms of that for the zeroth blade. The upwash velocity on the zeroth blade is related to its displacement

$$V(\xi,\pm 0)=-\left(ik-\frac{\partial}{\partial\xi}\right)W_0(\xi)\ \text{ for }\ \eta=0,0<\xi<2 \tag{9}$$

Each incremental blade section is assumed to undergo a rigid body motion

$$W_0=H_0+A_0(\xi-d_0) \tag{10}$$

where H_0, A_0, and d_0 are constants. H_0 represents the amplitude of a vertical displacement at $\xi=d_0$, and A_0 is the amplitude of the angular displacement about this point. Across the wake we require the pressure and the upwash velocity to be continuous. Far from the cascade all disturbances radiate outward.

Analytical Solution

The boundary condition (8) requires that the solution possess a certain blade-to-blade periodicity. The solution is assumed to satisfy the stronger periodicity condition

$$\theta(\xi+ms^+,\eta+mks^-)=e^{(im\sigma)}\theta(\xi,\eta) \tag{11}$$

where θ can denote any of the physical variables V or P.

Since disturbances do not propagate upstream in a transonic flow, any disturbances originating at or behind the Mach waves emanating from the trailing edges will not influence the flow upstream of these waves. Hence, the flow in the upstream region can be calculated independently of the flow in the downstream region. So, for the purpose of calculating the upstream flowfield, the blades can be extended downstream to infinity. The cascade is thus replaced by a row of semi-infinite plates. Let ϕ_1 denote the solution to this boundary value problem; then, in the region downstream of the trailing-edge Mach waves (Fig. 3), there must exist a function ϕ_2 such that $\phi=\phi_1+\phi_2$.

Upstream Solution

Transforming Eq. (4) by means of the following Fourier transform

$$\phi_1(\xi,\eta)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi^*(\alpha,\eta)e^{(-i\alpha\xi)}\mathrm{d}\alpha \tag{12}$$

yields

$$\frac{\partial^2\phi^*}{\partial\eta^2}+\phi^*\left(\alpha+\frac{k}{2}\right)\left(\frac{2}{k}\right)=0 \tag{13}$$

The solution to Eq. (13) is given by

$$\phi^*=A_1\exp[ik_1(\alpha+k_2)^{1/2}\eta]+B_1\exp[-ik_1(\alpha+k_2)^{1/2}\eta] \tag{14}$$

where $k_1=\sqrt{2/k}$ and $k_2=k/2$.

For the solution to be bounded, $A_1=0$ for $\eta<0$, and $B_1=0$ for $\eta>0$. Hence, $\phi_1(\xi,\eta)$ can be written as

$$\phi_1(\xi,\eta)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}h(\alpha)e^{(-i\alpha\xi)}\mathrm{sgn}(\eta)$$
$$\times\exp[\mathrm{sgn}(\eta)ik_1(\alpha+k_2)^{1/2}\eta]\mathrm{d}\alpha \tag{15}$$

Equation (15) is appropriate for an isolated blade. By summing the disturbances due to individual blades, we obtain the solution for a cascade of blades:

$$\phi_1(\xi,\eta)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty}h_m(\alpha)\exp[-i\alpha(\xi-ms^+)]$$
$$\times\mathrm{sgn}(\eta-mks^-)\exp[\mathrm{sgn}(\eta-mks^-)ik_1(\alpha+k_2)^{1/2}$$
$$\times(\eta-mks^-)]\mathrm{d}\alpha \tag{16}$$

In order to ensure that no waves propagate upstream and that the solution remains bounded at infinity (i.e., only outward propagating waves exist), the branch cut point associated with the branch point at $\alpha=-k_2$ is taken as shown in Fig. 4.

The signum function is defined as $\mathrm{sgn}\,\eta=\pm 1$ for $\eta\gtrless 0$ and used to produce a jump discontinuity in ϕ_1

$$[\phi_1(\xi)]_m=\lim_{\epsilon\to 0}[\phi_1(\xi,ms^-k+\epsilon)-\phi_1(\xi,mks^--\epsilon)]$$
$$=[\phi_1(\xi)]_m \tag{17}$$

across $\eta=kms^-$.

It is possible to satisfy the requirement that the upwash velocity be continuous while allowing for this discontinuity. ϕ_1 will satisfy the imposed periodicity condition if we let

$$h_m(\alpha)=e^{(im\sigma)}h_0(\alpha)\ \text{ for }\ m=0,\pm 1,\pm 2 \tag{18}$$

Inserting Eq. (18) into Eq. (16) yields

$$\phi_1=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}h_0(\alpha)\lambda_1(\alpha,\eta)e^{(-i\alpha\xi)}\mathrm{d}\alpha \tag{19}$$

$$\lambda_1(\alpha,\eta)=\frac{e^{[i(\sigma+\alpha s^+)]}\cos(\gamma_1\eta)-\cos[\gamma_1(\eta-ks^-)]}{2\sinh(\Delta_+)\sinh(\Delta_-)}$$
$$\text{for }\ 0<\eta<ks^- \tag{20}$$

$$\gamma_1=k_1(\alpha+k_2)^{1/2}$$
$$\Delta_+=(i/2)(\sigma+\alpha s^+-\gamma_1ks^-)$$
$$\Delta_-=(i/2)(\sigma+\alpha s^++\gamma_1ks^-) \tag{21}$$

Using Eq. (6) the upwash velocity V_1 can be expressed as

$$V_1(\xi,\eta)=\frac{\partial\phi_1}{\partial\eta}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}h_0(\alpha)K(\alpha,\eta)e^{(-i\alpha\xi)}\mathrm{d}\alpha \tag{22}$$

where

$$K(\alpha,\eta)=\frac{\partial\lambda_1}{\partial\eta}$$

$$=\frac{\gamma_1 k[e^{[i(\sigma+\alpha s^+)]}\sin(\gamma_1\eta)-\sin[\gamma_1(\eta-ks^-)]]}{2\sinh(\Delta_+)\sinh(\Delta_-)} \tag{23}$$

Since ϕ_1 satisfies the periodicity condition (11), it will be continuous across the lines $\eta=mks^-$, $-\infty<\xi<ms^+$ if it is continuous across the line, $\eta=0$, $-\infty<\xi<0$ passing through the $m=0$ blade. It follows from Eqs. (17) and (19) that this occurs if

$$\frac{2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}h_0(\alpha)e^{(-i\alpha\xi)}\mathrm{d}\alpha=0 \quad \text{for } \xi<0 \tag{24}$$

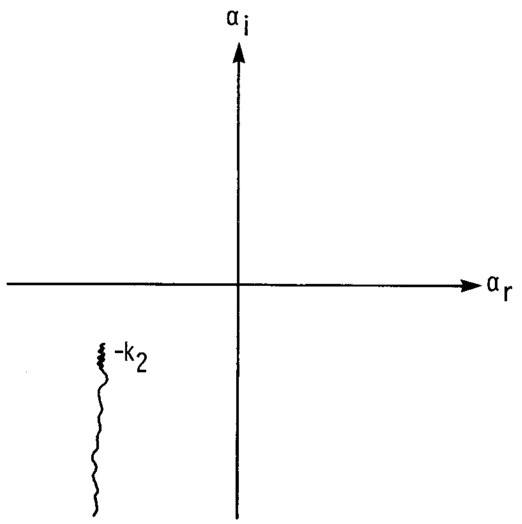

Fig. 4 Branch cut in a complex plane.

Assuming $h_0(\alpha)$ satisfies this equation, then ϕ_1, as defined by Eq. (19), will satisfy all of the imposed boundary conditions except that for the upwash at the blade surface. The upwash velocity V satisfies the periodicity conditions (11) and (8). It must also satisfy the conditions given by Eq. (9). By making use of Eq. (22), Eq. (9) can be written as

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}h_0(\alpha)K(\alpha,0)e^{(-i\alpha\xi)}\mathrm{d}\alpha=-\left(ik-\frac{\partial}{\partial\xi}\right)W_0(\xi) \tag{25}$$

Equations (24) and (25) constitute a set of dual integral equations that can be solved for $h_0(\alpha)$ by the Wiener-Hopf procedure. This procedure is outlined in Appendix A. By substituting the value of $h_0(\alpha)$ given by Eq. (A14) into Eq. (19), one obtains

$$\phi_1(\xi,\eta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\lambda_1(\alpha,\eta)}{K_-(-i\mu)K_+(\alpha)}\left[\frac{A'i+B'(\mathrm{d}/\mathrm{d}\xi)[\log K_-(\xi)]_{\xi=0}}{(\alpha+i\mu)}-\frac{B'}{(\alpha+i\mu)^2}\right]e^{(-i\alpha\xi)}\mathrm{d}\alpha \tag{26}$$

where μ is a positive small number set equal to zero after the evaluation of the contour integral. $K_\pm(\alpha)$ are nonzero analytic functions that have algebraic behavior at infinity in the upper and lower half-planes, respectively. The integral given by Eq. (26) can be evaluated using the method of residues. The resulting expression for ϕ_1 is

$$\phi_1=\frac{1}{2\pi}\left[A'i+B'\frac{\mathrm{d}}{\mathrm{d}\xi}[\log K_-(\mu)]_{\xi=0}\right](\bar{T}_1-\bar{\bar{T}}_1)-\frac{B}{2\pi}(\bar{T}_2-\bar{\bar{T}}_2) \tag{27}$$

where

For $0<\xi<s^+$

$$\bar{T}_1=2\pi ie^{i\sigma}\sum_{n=0}^{\infty}\frac{1}{\alpha_n^+}\frac{K_+(0)}{K_+(\alpha_n^+)}\frac{\cos(\gamma_1(\alpha_n^+)\eta)\exp[-i\alpha_n^+(\xi-s^+)]}{2K(0,0)i\sin(\alpha_n^+s^++\sigma)(\mathrm{d}\Delta_+(\alpha_n^+)/\mathrm{d}\alpha)}$$
$$+2\pi ie^{i\sigma}\sum_{n=1}^{\infty}\frac{1}{\alpha_{-n}^+}\frac{K_+(0)}{K_+(\alpha_{-n}^+)}\frac{\exp[-i\alpha_{-n}^+(\xi-s^+)]\cos[\gamma_1(\alpha_{-n}^+)\eta]}{2K(0,0)i\sin(\alpha_{-n}^+s^++\sigma)(\mathrm{d}\Delta_+(\alpha_{-n}^+)/\mathrm{d}\alpha)} \tag{28}$$

where

$$\frac{\mathrm{d}\Delta_+(\alpha_{\pm n}^+)}{\mathrm{d}\alpha}=\frac{i}{2}\left[s^+-\frac{(k_1ks^-)^2}{2(\alpha_{\pm n}^+s^+-\Gamma_{\pm n})}\right]$$

For $\xi>s^+$

$$\bar{T}_1=\frac{2\pi i\cos(\eta)e^{i\sigma}}{k\sin(ks^-)}+2\pi i\sum_{n=0}^{\infty}\frac{\exp[i(a_ns^++\sigma)]}{a_n}\frac{K_-(a_n)}{K_-(0)}\frac{\cos(n\pi/ks^-)\eta\exp(-ia_n\xi)}{ks^-[\delta_{0,n}+(-1)^n]} \tag{29}$$

where $a_n=-k/2+(n\pi/ks^-)^2$

For $\xi>0$

$$\bar{\bar{T}}_1=\frac{2\pi i\cos(\eta-ks^-)}{k\sin(ks^-)}+2\pi i\sum_{n=0}^{\infty}\frac{K_-(a_n)}{K_-(0)}\frac{\cos[(n\pi/ks^-)(ks^--\eta)]\exp(-ia_n\xi)}{a_nks^-[\delta_{0,n}+(-1)^n]} \tag{30}$$

For $0<\xi<s^+$

$$\bar{T}_2=2\pi i\sum_{n=0}^{\infty}\frac{e^{(i\sigma)}}{\alpha_n^{+2}}\frac{K_+(0)}{K_+(\alpha_n^+)}\frac{\cos[\gamma_1(\alpha_n^+)\eta]\exp[-i\alpha_n^+(\xi-s^+)]}{2K(0,0)i\sin(\alpha_n^+s^++\sigma)(\mathrm{d}\Delta_+(\alpha_n^+)/\mathrm{d}\alpha)}$$

$$+2\pi i\sum_{n=1}^{\infty}\frac{e^{(i\sigma)}}{\alpha_{-n}^{+2}}\frac{K_+(0)}{K_+(\alpha_{-n}^+)}\frac{\cos[\gamma_1(\alpha_{-n}^+)\eta]\exp[-i\alpha_{-n}^+(\xi-s^+)]}{2K(0,0)i\sin(\alpha_{-n}^+s^++\sigma)(\mathrm{d}\Delta_+(\alpha_{-n}^+)/\mathrm{d}\alpha)} \tag{31}$$

For $\xi>s^+$

$$\bar{T}_2=2\pi i\frac{e^{(i\sigma)}\cos(\eta)}{k\sin(ks^-)}\frac{\mathrm{d}}{\mathrm{d}\alpha}\{\log K_-(\alpha)\}_{\alpha=0}-2\pi i\frac{e^{(i\sigma)}\cos(\eta)}{k\sin(ks^-)}(s^+-\xi)-\frac{2\pi i\eta e^{(i\sigma)}}{k^2\sin(ks^-)}\sin(\eta)$$

$$-\frac{2\pi i\cos(\eta)e^{(i\sigma)}}{k^2\sin^2(ks^-)}[\sin(ks^-)+ks^-\cos(ks^-)]+2\pi i\sum_{n=0}^{\infty}\frac{K_-(a_n)}{K_-(0)}\frac{e^{(-ia_n\xi)}}{a_n^2}\frac{\cos(n\pi/ks^-)\eta\exp[i(\sigma+a_ns^+)]}{ks^-[\delta_{0,n}+(-1)^n]} \tag{32}$$

For $\xi>0$

$$\bar{\bar{T}}_2=2\pi i\frac{\cos(ks^--\eta)}{k\sin(ks^-)}\frac{\mathrm{d}}{\mathrm{d}\alpha}[\log K_-(\alpha)]_{\alpha=0}+2\pi\xi\frac{\cos(ks^--\eta)}{k\sin(ks^-)}-2\pi i\frac{(ks^--\eta)\sin(ks^--\eta)}{k^2\sin(ks^-)}$$

$$-2\pi i\frac{\cos(ks^--\eta)}{k^2\sin^2(ks^-)}[\sin(ks^-)+ks^-\cos(ks^-)]+2\pi i\sum_{n=0}^{\infty}\frac{K_-(a_n)}{K_-(0)}\frac{e^{-ia_n\xi}}{a_n^2}\frac{\cos[(n\pi/ks^-)(ks^--\eta)]}{ks^-[\delta_{0,n}+(-1)^n]} \tag{33}$$

By substituting Eqs. (28-33) into Eq. (27), the perturbation potential ϕ_1 can be obtained.

Downstream Solution

The perturbation potential ϕ_1 does not represent the solution to the problem in the region downstream of the trailing-edge Mach wave. It must be augmented by a solution ϕ_2, which satisfies the following boundary conditions (see Fig. 3):

$$V_2(\xi^*+ms^+,mks^-)=0 \text{ for } -\infty<\xi^*<0 \qquad [P_2(\xi^*)]_m=-[P_1(\xi^*)]_m \text{ for } 0<\xi^*<\infty \tag{34}$$

where $\xi^*=\xi-2$ and $[P(\xi^*)]_m$ denotes the jump in pressure, i.e.,

$$P(\xi^*+ms^+,mks^-+0)-P(\xi+ms^+,mks^--0)$$

Then $\phi=\phi_1+\phi_2$ will satisfy the correct boundary conditions on the surface of each blade and across the wakes. The constructed solution, however, will not necessarily satisfy the correct conditions across the horizontal lines extending upstream to infinity from the leading edge of each blade. These conditions will be satisfied if ϕ_2 is identically zero in the region upstream of the trailing-edge Mach waves. In order to determine ϕ_2, $[p_1(\xi^*)]_0$ across the wake of the zeroth blade must first be calculated. This can be accomplished by using Eqs. (7) and (27). The resulting expression is

$$[P_2(\xi^*)]=-[P_1(\xi^*)]=P_1'+P_1''\xi^*+\sum_{n=0}^{\infty}P_n^*e^{(-ia_n\xi^*)} \tag{35}$$

where

$$P_1'=2\frac{[\cos\sigma-\cos(ks^-)]}{k\sin(ks^-)}\{ik(A'+2B')+B'ks^-\cot(ks^-)\}-2B's^-+\frac{2B's^+\sin(\sigma)}{\sin(ks^-)}$$

$$P_1''=\frac{2B'i}{\sin(ks^-)}[\cos(\sigma)-\cos(ks^-)]$$

$$P_n^*=-2i\frac{Q_n}{a_n^2}\frac{K_-(a_n)}{K_-(0)}\frac{(k+a_n)e^{(-2ia_n)}\{\cos(a_ns^++\sigma)-(-1)^n\}}{ks^-\{\delta_{0,n}+(-1)^n\}}$$

$$Q_n=-A'a_n+B'ia_n\left\{\frac{\mathrm{d}}{\mathrm{d}\xi^*}[\log K_-(\xi^*)]\right\}_{\xi^*=0}-B'i \tag{36}$$

Equations (35) and (36) suggest that a solution of the form

$$\phi_2=\tilde{\phi}+\sum_{n=0}^{\infty}\phi_{\pm}^{(n)} \tag{37}$$

is required to ensure that ϕ_2 represents a solution of Eq. (1) that satisfies the periodicity condition (11) as well as the radiation conditions. These variables are defined as

$$\phi_{\pm}^{(n)}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h_n^{\pm}(\alpha)\lambda_1(\alpha,\eta)e^{(-i\alpha\xi^*)}\mathrm{d}\alpha \tag{38}$$

$$\tilde{\phi}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{h}(\alpha)\lambda_1(\alpha,\eta)e^{(-i\alpha\xi^*)}\mathrm{d}\alpha \tag{39}$$

Using boundary conditions given by Eq. (34), Eqs. (38) and (39) lead to a set of dual integral equations for $h_n^{\pm}$ and $\tilde{h}$. These equations can be solved in a fashion similar to that outlined in Appendix A.

One can show that $\phi_{\pm}^{(n)}$ and $\tilde{\phi}$ can be written as

$$\phi_{\pm}^{(n)}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{G_n^{\pm}e^{(-i\alpha\xi^*)}}{(\alpha+k)}K_+(\alpha)\frac{\exp[i(\sigma+\alpha s^+)]\cos(\gamma_1\eta)-\cos[\gamma_1(\eta-ks^-)]}{\gamma_1 k\sin(\gamma_1 ks^-)}\mathrm{d}\alpha \tag{40}$$

$$\tilde{\phi}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\tilde{G}_+e^{-i\alpha\xi^*}}{(\alpha+k)}K_+(\alpha)\frac{\exp[i(\sigma+\alpha s^+)]\cos(\gamma_1\eta)-\cos[\gamma_1(\eta-ks^-)]}{\gamma_1 k\sin(\gamma_1 ks^-)}\mathrm{d}\alpha \tag{41}$$

where

$$G_{n+}^{\pm}=-\frac{1}{2\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{P_n^*}{(\alpha-a_n)K'_-(a_n)}$$

$$\tilde{G}_+(\alpha)=-\frac{1}{2\sqrt{2\pi}}\,\frac{P_1'}{(\alpha+i\mu)K'_-(-i\mu)}+\frac{iP_1''}{2\sqrt{2\pi}(\alpha+i\mu)}\,\frac{\{(\mathrm{d}/\mathrm{d}\xi)[\log K'_-(\xi^*)]\}_{\xi^*=-i\mu}}{K'_-(-i\mu)}-\frac{iP_1''}{2\sqrt{2\pi}i(\alpha+i\mu)^2K'_-(-i\mu)} \tag{42}$$

The method of residues is used to evaluate the contour integrals. For points in the region upstream of the Mach wave emanating from the trailing edge, the integration contour is closed in the upper half-plane. But since all the poles lie in the lower half-plane, $\phi_{\pm}^{(n)}$ and $\tilde{\phi}$ are identically zero in the upstream region. This proves that $\phi=\phi_1+\phi_2$ is the correct solution to the stated boundary value problem. By closing the appropriate integration contours, $\phi_{\pm}^{n}$ and $\tilde{\phi}$ are found to be

$$\begin{aligned}\phi_{\pm}^{n}=&-\frac{s^-i}{2}\,\frac{P_n^*\exp(-ia_n\xi^*)\cos[(n\pi/ks^-)(ks^--\eta)]}{C'(a_n)(k+a_n)}-\frac{s^-i}{2}\,\frac{P_n^*C_-(-k)e^{(ik\xi^*)}}{C'(-k)(k+a_n)C_-(a_n)}\cos[i(\eta-ks^-)]\\&-\frac{s^-i}{2}\sum_{m=0}^{\infty}\frac{P_n^*\exp(-i\alpha_m^-\xi^*)C_-(\alpha_m^-)\cos[\gamma_1(\alpha_m^-)(\eta-ks^-)]}{(\alpha_m^--a_n)C_-(a_n)(k+\alpha_m^-)2s^-i\sin(\alpha_m^-s^++\sigma)(\mathrm{d}\Delta_-(\alpha_m^-)/\mathrm{d}\alpha)}\\&-\frac{s^-i}{2}\sum_{m=1}^{\infty}\frac{P_n^*\exp(-i\alpha_{-m}^-\xi^*)C_-(\alpha_{-m}^-)\cos[\gamma_1(\alpha_{-m}^-)(\eta-ks^-)]}{(\alpha_{-m}^--a_n)(k+\alpha_{-m}^-)2s^-i\sin(\alpha_{-m}^-s^++\sigma)(\mathrm{d}\Delta_-(\alpha_{-m}^-)/\mathrm{d}\alpha)C_-(a_n)}\end{aligned} \tag{43}$$

where $C'(\alpha)=2\sinh(\Delta_+)\sinh(\Delta_-)$, $C_-(\alpha)=K'_-(\alpha)$.

$$\begin{aligned}\tilde{\phi}=&-\frac{is^-}{2}\left[\frac{\cos(\eta-ks^-)}{k^2C'(0)}-\frac{C(-k)\cos[i(\eta-ks^-)]e^{ik\xi^*}}{k^2C_-(0)C'_-(-k)}\right](kP_{11}-iP_1'')+\frac{s^-P_2}{2kC'(0)}\left[\cos(\eta-ks^-)\left\{\frac{\mathrm{d}}{\mathrm{d}\alpha}(\log C_-(\alpha))_{\alpha=0}\right.\right.\\&\left.\left.-\frac{\mathrm{d}}{\mathrm{d}\alpha}(\log C'(\alpha))_{\alpha=0}-i\xi^*\right\}-\frac{(\eta-ks^-)}{k}\sin(\eta-ks^-)\right]-\frac{is^-}{2}\sum_{n=0}^{\infty}\frac{C_-(\alpha_n^-)\cos[\gamma_1(\alpha_n^-)(\eta-ks^-)]\exp(-i\alpha_n^-\xi^*)[\alpha_n^-P_{11}+iP_1'']}{\alpha_n^{-2}(\alpha_n^-+k)C_-(0)2s^-i\sin(\alpha_n^-s^++\sigma)(\mathrm{d}\Delta_-(\alpha_n^-)/\mathrm{d}\alpha)}\\&-\frac{is^-}{2}\sum_{n=1}^{\infty}\frac{C_-(\alpha_{-n}^-)\cos[\gamma_1(\alpha_{-n}^-)(\eta-ks^-)]\exp(-i\alpha_{-n}^-\xi^*)[\alpha_{-n}^-P_{11}+iP_1'']}{\alpha_{-n}^{-2}(\alpha_{-n}^-+k)C_-(0)2s^-i\sin(\alpha_{-n}^-s^++\sigma)(\mathrm{d}\Delta_-(\alpha_{-n}^-)/\mathrm{d}\alpha)}\end{aligned} \tag{44}$$

By substituting values of $\phi_{\pm}^{(n)}$ and $\tilde{\phi}$ given by Eqs. (43) and (44) into Eqs. (37), ϕ_2 can be readily evaluated. ϕ_1 is given by Eq. (27); thus, the total solution ϕ can be obtained by adding ϕ_1 and ϕ_2.

The series expressions for ϕ_1 and ϕ_2 are absolutely and uniformly convergent and, hence, ϕ is a continuous function of ξ. The expression given for pressure can be obtained by substituting ϕ into Eq. (7). Differentiating ϕ with respect to ξ leads to a divergent series.[15] A similar observation was made by Savkar[14] in his analysis. The unsteady lift acting on the reference blade can be calculated by integrating the pressure difference across the blade. This integration leads to an expression for lift in terms of the perturbations potential. Similarly, the moment acting at the center-of-reference airfoil can also be expressed in terms of the perturbation velocity potential. Since the series expressions for ϕ are absolutely convergent, expressions for lift and moment are also absolutely convergent. The relevant expressions are

$$L = ik\int_0^2 \Delta\phi \mathrm{d}\xi - [\Delta\phi]_0^2 \tag{45}$$

$$M = ik\int_0^2 \Delta\phi\xi \mathrm{d}\xi + \int_0^2 \Delta\phi \mathrm{d}\xi - [\xi\Delta\phi]_0^2 \tag{46}$$

where

$$\Delta\phi = \{\phi\}_{\eta=0} - \{\phi\}_{\eta=ks} - e^{-i\sigma} \tag{47}$$

Results and Discussion

A computer code was developed to obtain the unsteady aerodynamic forces acting on cascade blades. This code gives the results for both bending and torsional motions. The input parameters needed for the code are reduced frequency, stagger angle pitch-to-chord ratio, interblade phase angle, and pitching axis location.

A transonic similarity law can be used to compare the results of the present analysis to those from linearized unsteady subsonic and supersonic cascade theories. The following requirements need to be satisfied to make use of the similarity transformation:

$$k \gg |1 - M_1|, \qquad k \gg \delta^{2/3}$$

The derivation of this similarity law can be found in Appendix B.

Figures 5 and 6 show the variation of $|L/(\pi k^2)|$ and $|M/(\pi k^2)|$ [refer to Eqs. (45) and (46)] for various values of reduced frequency.

By making use of the transonic similarity law, the present results are compared to the results obtained from Smith's[2] analysis for a Mach number = 0.9 in Fig. 5 and to Adamczyk and Goldstein's[3] analysis for a Mach number = 1.1 in Fig. 6. The parameters chosen for comparison are: stagger angle = 30 deg, solidity = 1.0, interblade phase angle = 0 deg, mode = torsion, pitching axis = midchord.

It is observed that the present results are in good agreement with the subsonic and supersonic results for high reduced frequencies. Similar results were obtained over a range of cascade geometry parameters.

For bending motion, the nondimensional work per cycle done by the flow on the blades is equal to $\pi H_0 \mathrm{Im}(L)$. When this quantity is positive, the blade receives the energy from the flow and becomes unstable. Hence, the cascade will flutter when the imaginary part of L, $\mathrm{Im}(L)$ is positive. If $\mathrm{Im}(L)$ is negative, the blades lose energy to the flow and are thus stable. This stability criterion was examined over a range of reduced frequencies, stagger angles, solidities, and locations of pitching axis. We found that the imaginary part of the lift for pure bending motion always remained negative. Hence, we concluded that a lightly loaded cascade would not encounter bending flutter at transonic speeds. This observation is consistent with the results obtained by Savkar[14] for an airfoil oscillating in a wind tunnel for transonic flow and with Adamczyk and Goldstein[3] for the case of supersonic flow at low levels of aerodynamic loading.

For torsional motion, the nondimensional work per cycle done by the flow on the blades is equal to $\pi A_0 \mathrm{Im}(M)$. Once again, the cascade becomes unstable if this quantity is

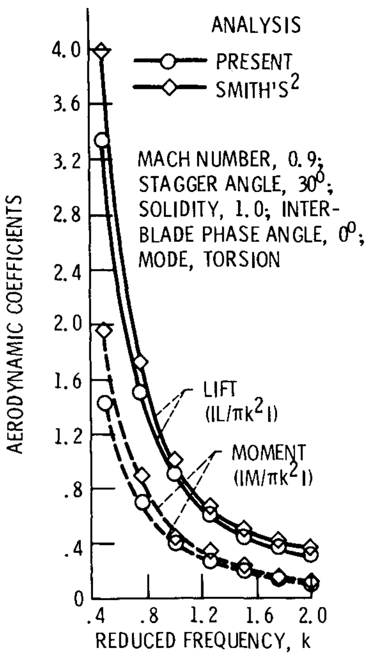

Fig. 5 Comparison of present analysis results with Smith's[2] analysis results.

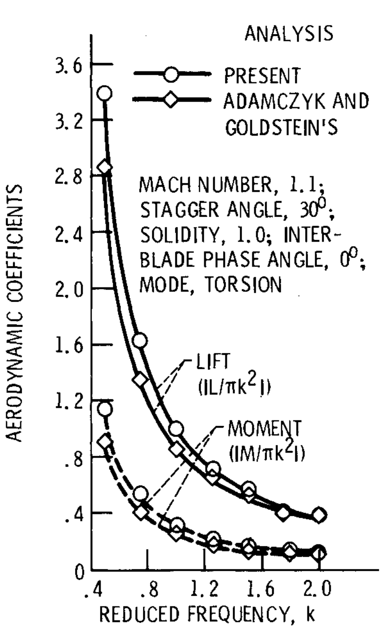

Fig. 6 Comparison of present analysis results with Adamczyk and Goldstein[3] analysis results.

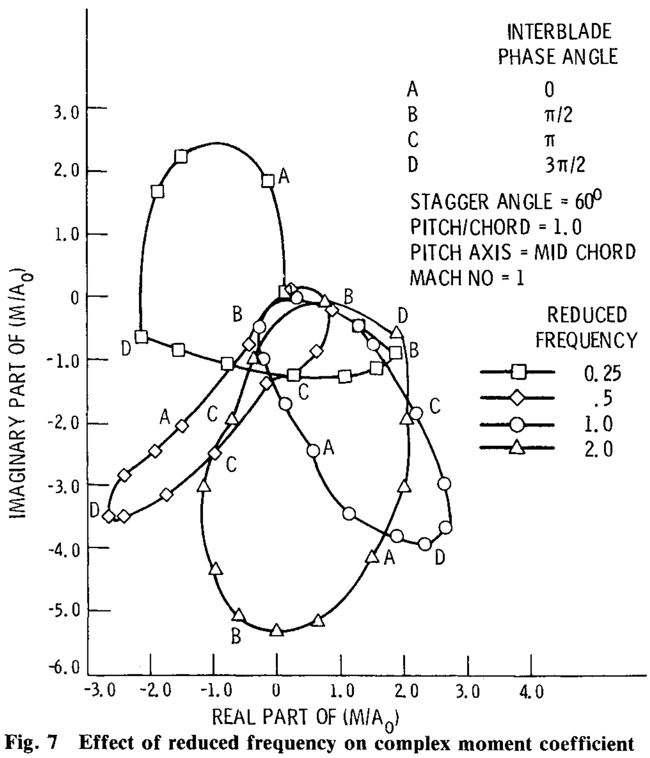

Fig. 7 Effect of reduced frequency on complex moment coefficient for pitching motion about center of airfoil.

positive. A series of computations were performed to examine the dependence of this parameter on cascade geometry and reduced frequency.

The geometry of the cascade assumed in this study is solidity = 1.0 and stagger angle = 60 deg, unless otherwise noted. The pitching axis is assumed to lie at midchord. In Fig. 7, the real and imaginary parts of moment are plotted as a function of reduced frequency. It is observed that the blade row tends to become more stable as the reduced frequencies are increased. A similar observation is made by Adamczyk et al.[3] for the case of supersonic flow. Figure 8

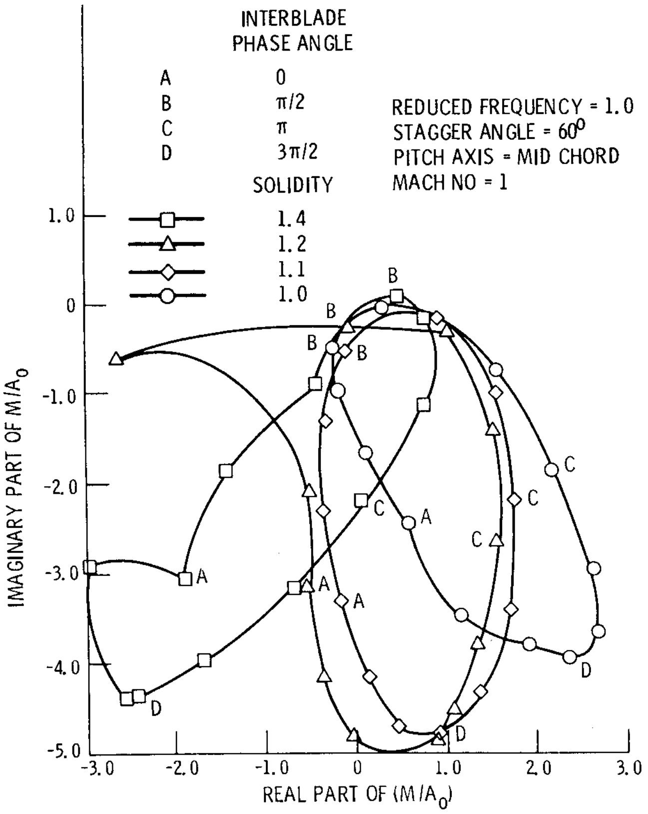

Fig. 8 Effect of cascade solidity on complex moment coefficient for pitching motion about center of airfoil.

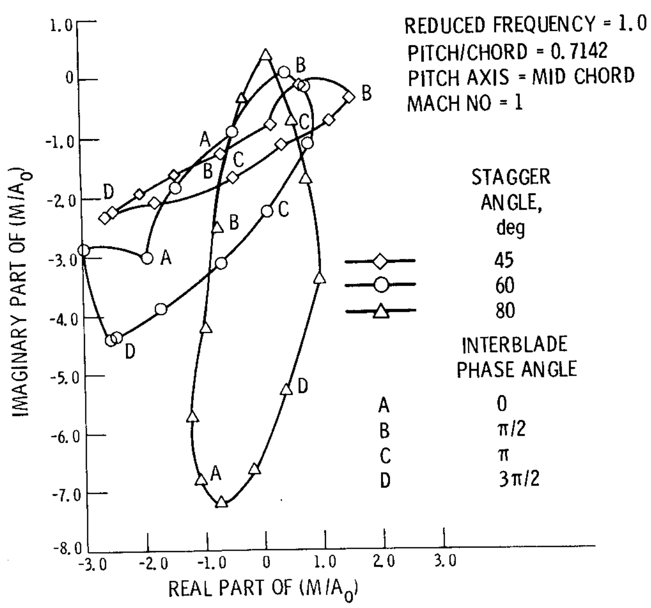

Fig. 9 Effect of cascade stagger angle on complex moment coefficient for pitching motion about center of airfoil.

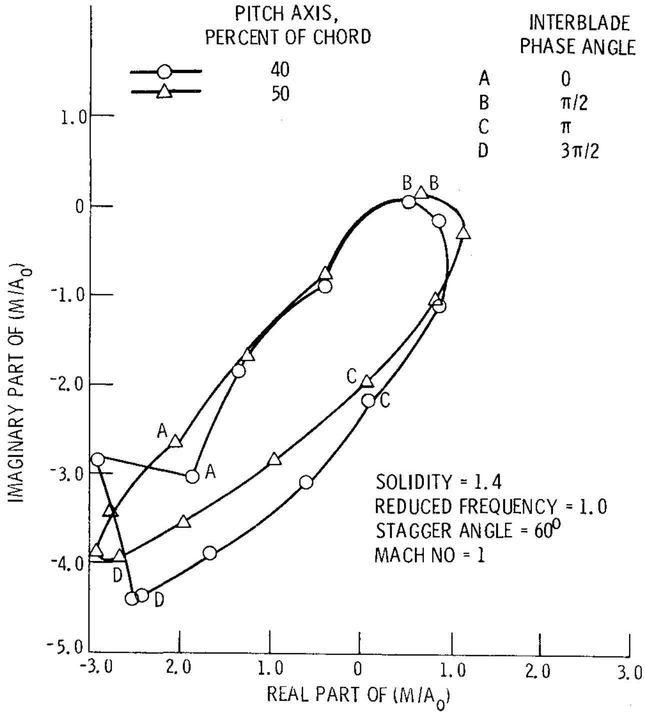

Fig. 10 Effect of pitching axis on complex moment coefficient for pitching motion about center of airfoil.

shows the effect of solidity of the blade row on stability. Increasing the cascade solidity has a slight destabilizing effect on the cascade. Stagger angle also plays an important role in setting the stability boundary. Its influence on stability is shown in Fig. 9. It is noted that the stability of the cascade can be increased by reducing the stagger angle. The effect of pitching axis location on stability is shown in Fig. 10. It is observed that a rearward shift of pitching axis decreases the stability of the cascade. Similar trends have been reported in Ref. 3.

Concluding Remarks

An exact analytical solution was obtained for unsteady linearized transonic cascade flow problems at Mach number = 1.0. By making use of a transonic similarity law, this analysis could be used for subsonic and supersonic cascade flow problems. It was observed that a cascade does not experience instability for the bending mode. It was shown that increasing the reduced frequency and decreasing the stagger angle and solidity had a stabilizing effect on torsional flutter. These trends are consistent with those predicted by linearized supersonic unsteady cascade analysis. The aerodynamic model developed in the present analysis can be coupled with a structural model, allowing one to study the aeroelastic response of a cascade of blades in transonic flows.

Appendix A

The Weiner-Hopf procedure for solving the first set of integral equations associated with this work will be outlined. These equations are of the form

$$\frac{2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h_0(\alpha)e^{(-i\alpha\xi)}\mathrm{d}\alpha = 0 \quad \text{for } \xi < 0 \tag{24}$$

$$V_1(\alpha) = \frac{\partial\phi}{\partial\eta} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h_0(\alpha)K(\alpha,0)e^{(-i\alpha\xi)}\mathrm{d}\alpha \tag{25}$$

From Eq. (24) it is apparent that $h_0(\alpha)$ must be analytic in the upper half-plane. We define $h_0(\alpha)$ accordingly,

$$h_0(\alpha)=H_+(\alpha) \tag{A1}$$

By taking the inverse transform of Eq. (25), we get

$$h_0(\alpha)K(\alpha,0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} V_1(\alpha)e^{(i\alpha\xi)}d\xi \tag{A2}$$

Let

$$V_1(\alpha)=V_1'(\xi,0)+V_1''(\xi,0) \tag{A3}$$

where

$$V_1'(\xi,0)=0 \text{ for } \xi\le 0, \quad V_1''(\xi,0)=0 \text{ for } \xi\ge 0$$

$$V_+(\alpha)=\frac{1}{\sqrt{2\pi}}\int_0^{\infty} V_1'(\xi,0)e^{(i\alpha\xi)}d\xi$$

$$V_-(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} V_1''(\xi,0)e^{(i\alpha\xi)}d\xi \tag{A4}$$

Note that $V_+(\alpha)$ and $V_-(\alpha)$ are analytic in the upper and lower half-planes, respectively. Next, $K(\alpha,0)$ is factorized into the product

$$K(\alpha,0)=K_+(\alpha)K_-(\alpha) \tag{A5}$$

where $K_+(\alpha)$ is analytic, nonzero, and bounded in the upper half-plane and $K_-(\alpha)$ is analytic, nonzero, and bounded as $\alpha\to\infty$ in the lower half-plane.

By substituting Eqs. (A1), (A4), and (A5) into Eq. (A2), one obtains

$$\frac{V_+(\alpha)}{K_-(\alpha)}+\frac{V_-(\alpha)}{K_-(\alpha)}=H_+(\alpha)K_+(\alpha) \tag{A6}$$

Next, the quotient $V_+(\alpha)/K_-(\alpha)$ is factorized into the form

$$V_+(\alpha)/K_-(\alpha)=G_+(\alpha)+G_-(\alpha) \tag{A7}$$

where $G_+(\alpha)$ is analytic and has algebraic behavior at infinity in the upper half-plane and $G_-(\alpha)$ is analytic, having algebraic behavior at infinity in the lower half-plane.

Equations (A6) can be rewritten as

$$G_+(\alpha)-H_+(\alpha)K_+(\alpha)=-G_-(\alpha)-[V_-(\alpha)/K_-(\alpha)] \tag{A8}$$

The left-hand side of this equation is analytical in the upper half-plane, the right-hand side is analytic in the lower half-plane. Therefore, these two functions are analytic continuations of one another and together define an analytic function. By using the known relations between asymptotic expansions for large α of various Fourier transforms and the behavior of the physical variables near $\xi=0$, it can be shown that the latter quantities will remain bounded at $\xi=0$ if the left- and right-hand sides of this equation vanish in their appropriate planes. By making use of Liouville's theorem, $h_0(\alpha)$ can be written as

$$h_0(\alpha)=H_+(\alpha)=G_+(\alpha)/K_+(\alpha) \tag{A9}$$

By substituting Eqs. (10) into Eq. (9), the upwash velocity on the zeroth blade is

$$V=[A'+B'\xi]e^{-\mu\xi} \tag{A10}$$

where

$$A'=(1+ikd_0)A_0-ikH_0, \quad B'=-ikA_0, \quad 0<\mu\ll 1$$

By substituting Eq. (A10) into Eq. (A4), $V_+(\alpha)$ can be determined as

$$V_+(\alpha)=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}(A'+B'\xi)e^{[i(\alpha+i\mu)\xi]}d\alpha \tag{A11}$$

which further reduces to

$$V_+(\alpha)=\frac{A'i}{\sqrt{2\pi}(\alpha+i\mu)}-\frac{B'}{\sqrt{2\pi}(\alpha+i\mu)^2} \tag{A12}$$

$G_+(\alpha)$ can be determined by inserting Eqs. (A12) into Eq. (A7).

$$G_+(\alpha)=\frac{A'i+B'(d/d\xi)[\log(K_-(\xi))]_{\xi=0}}{\sqrt{2\pi}(\alpha+i\mu)K_-(-i\mu)}-\frac{B'}{\sqrt{2\pi}K_-(-i\mu)(\alpha+i\mu)^2} \tag{A13}$$

$h_0(\alpha)$ can then be evaluated by substituting Eq. (A13) into Eq. (A9).

$$h_0(\alpha)=\frac{A'i+B'(d/d\xi)[\log(K_-(\xi))]_{\xi=0}}{\sqrt{2\pi}(\alpha+i\mu)K_-(-i\mu)K_+(\alpha)}-\frac{B'}{\sqrt{2\pi}(\alpha+i\mu)^2K_-(-i\mu)K_+(\alpha)} \tag{A14}$$

Appendix B

The linearized unsteady transonic flow equations neglecting higher-order terms is given as[9]

$$k^2\frac{\partial^2\bar\phi}{\partial\eta^2}-2M_1^2\frac{\partial^2\bar\phi}{\partial\xi\partial\tau}-M_1^2\frac{\partial^2\bar\phi}{\partial\tau^2} \tag{B1}$$

The Mach number dependence in the equation can be eliminated by a Prandtl-Glauert type of transformation: Let

$$\xi'=\lambda_\xi\xi, \quad \eta'=\lambda_\eta\eta, \quad \bar\phi(\xi_1',\eta',\tau')=\lambda_\phi\bar\phi(\xi,\eta,\tau) \tag{B2}$$

where λ_ξ, λ_η, and λ_ϕ are scaling factors.

Then

$$\frac{\partial^2\bar\phi}{\partial\xi\partial\tau}=\frac{\lambda_\xi}{\lambda_\phi}\frac{\partial^2\bar\phi_1}{\partial\xi'\partial\tau'}, \quad \frac{\partial^2\bar\phi}{\partial\eta^2}=\frac{\lambda_\eta^2}{\lambda_\phi}\frac{\partial^2\bar\phi_1}{\partial\eta'^2}$$

$$\frac{\partial^2\bar\phi}{\partial\tau^2}=\frac{1}{\lambda_\phi}\frac{\partial^2\bar\phi_1}{\partial\tau'^2} \tag{B3}$$

Substituting Eq. (B3) into Eq. (B1) yields

$$\frac{\lambda_\eta^2}{\lambda_\phi}k^2\frac{\partial^2\bar\phi_1}{\partial\eta'^2}-2M_1^2\frac{\lambda_\xi}{\lambda_\phi}\frac{\partial^2\bar\phi_1}{\partial\xi'\partial\tau'}-\frac{M_1^2}{\lambda_\phi}\frac{\partial^2\bar\phi_1}{\partial\tau'^2} \tag{B4}$$

Assuming

$$\lambda_\eta=M_1, \quad \lambda_\xi=1, \quad \lambda_\phi=1 \tag{B5}$$

Eq. (B4) can be rewritten as

$$k^2\frac{\partial^2\bar\phi_1}{\partial\eta'^2}-\frac{2\partial^2\bar\phi_1}{\partial\xi'\partial\tau'}-\frac{\partial^2\bar\phi_1}{\partial\tau'^2}=0 \tag{B6}$$

which is independent of Mach number.

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